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Author(s): IAN D. MORRIS
Article Title: The Mañé–Conze–Guivarc'h lemma for intermittent maps of the circle
Year of publication: 2009
Link to published version: http://dx.doi.org/10.1017/S0143385708000837
Publisher statement: None

Ergod. Th. & Dynam. Sys. (First published online 2009), page 1 of 9*
doi:10.1017/S0143385708000837 © *2009 Cambridge University Press*
*Provisional—final page numbers to be inserted when paper edition is published

The Mañé–Conze–Guivarc'h lemma for intermittent maps of the circle

IAN D. MORRIS

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK
(e-mail: Ian.Morris@warwick.ac.uk)

(*Received* 2 *October* 2007 *and accepted in revised form* 23 *July* 2008)

Abstract. We study the existence of solutions g to the functional inequality $f \leq g \circ T - g + \beta$, where f is a prescribed continuous function, T is a weakly expanding transformation of the circle having an indifferent fixed point, and β is the maximum ergodic average of f. Using a method due to T. Bousch, we show that continuous solutions g always exist when the Hölder exponent of f is close to 1. In the converse direction, we construct explicit examples of continuous functions f with low Hölder exponent for which no continuous solution g exists. We give sharp estimates on the best possible Hölder regularity of a solution g given the Hölder regularity of f.

1. *Introduction*

Let $T: X \to X$ be a discrete dynamical system, and let $\mathcal{M}_T$ be the set of all Borel probability measures which are invariant under the map T. For a given continuous function $f: X \to \mathbb{R}$, we define the maximum ergodic average $\beta(f)$ by

$$\beta(f) = \sup_{\mu \in \mathcal{M}_T} \int f \, d\mu,$$

and say that $\nu \in \mathcal{M}_T$ is a *maximizing measure* for f if it satisfies $\int f \, d\nu = \beta(f)$. The study of maximizing measures has recently become the focus of significant research interest. While early articles of Bousch and Jenkinson [**2, 14**] were motivated by abstract questions concerning the geometric structure of the set of measures $\mathcal{M}_T$, questions relating to maximizing measures have also appeared in research into chaotic control [**13, 25**], Livšic-type theorems [**6**], thermodynamic formalism [**9, 15, 16**], Tetris heaps [**7**], and the Lagarias–Wang finiteness conjecture in linear algebra [**7**].

This article is concerned with a key technical tool that arises in the study of maximizing measures, which we call the *Mañé–Conze–Guivarc'h lemma*. A lemma of this type takes the following form: given a continuous function $f: X \to \mathbb{R}$ with some prescribed regularity, under suitable dynamical hypotheses there exists a continuous function

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$g: X \to \mathbb{R}$ with the property that $f \le g \circ T - g + \beta(f)$. This relation is equivalent to the statement that there exists a continuous g such that $\sup(f + g - g \circ T) = \beta(f)$. Conze and Guivarc'h's version of this lemma may be found in the unpublished manuscript [**10**]. It has been noted that theorems of a similar character occur in the field of optimal control, e.g. [**1, 17**]; this relationship is examined in Bousch's preprint [**5**].

We briefly describe the immediate implications of this result. First, let us rewrite the aforementioned inequality in the form $f = g \circ T - g + \beta(f) - r$, where r is continuous and satisfies $r \ge 0$. We then obtain $\int f\,d\nu = \beta(f) - \int r\,d\nu$ for every $\nu \in \mathcal{M}_T$, and so ν is maximizing for f if and only if $\int r\,d\mu = 0$. Since $r(x) \ge 0$ for all x, we conclude that the maximizing measures of f are precisely those invariant measures ν whose support lies in the compact set $r^{-1}(0)$. This leads to the *subordination principle* described by Bousch [**3**]: if invariant measures μ, ν satisfy $\operatorname{supp} \nu \subseteq \operatorname{supp} \mu$ and μ is a maximizing measure for f, then the 'subordinate' measure ν is maximizing also. It has been shown that this subordination principle can fail to hold when the regularity of f is relaxed [**6**].

A particularly interesting application of the Mañé–Conze–Guivarc'h lemma is a recent result of Bousch [**4**] which shows that for dynamical systems satisfying a Mañé–Conze–Guivarc'h lemma, measures supported on periodic orbits are the only maximizing measures that persist under Lipschitz perturbations of the observable f. A similar result was previously shown by G. Yuan and B. R. Hunt under more restrictive dynamical assumptions [**25**]. Mañé–Conze–Guivarc'h-type lemmas have also been found useful in circumstances that are not *a priori* related to maximizing measures [**20**].

When $T: X \to X$ is an expanding map, a subshift of finite type or an Anosov diffeomorphism, and $f: X \to \mathbb{R}$ is Hölder continuous, it is known that we can always find $g: X \to \mathbb{R}$ Hölder continuous such that $f \le g \circ T - g + \beta(f)$ is satisfied [**3, 11, 19, 22**]. The purpose of the present article is to examine the extension of this result to a simple class of non-uniformly hyperbolic dynamical systems on the circle, namely the case in which T is uniformly expanding except in the neighbourhood of a weakly repelling fixed point.

Previously, it was shown by Souza [**23**] that for an expanding map $T: [0, 1] \to [0, 1]$ with a weakly repelling fixed point, a Mañé–Conze–Guivarc'h lemma can be proved when f is Hölder continuous and monotone in some neighbourhood of the indifferent fixed point z, and additionally satisfies $\int f\,d\nu_- < f(z) < \int f\,d\nu_+$ for some $\nu_-, \nu_+ \in \mathcal{M}_T$. Prior to the research described in this article, S. Branton had shown that when f is Lipschitz continuous, Souza's conditions may be removed [**8**]. In this article, using a different method from that of S. Branton, we study the case in which f is Hölder and prove a complementary result showing that solutions can fail to exist in certain situations where f is Hölder continuous with exponent close to 0.

Let $\mathbb{T} = \mathbb{R} \bmod \mathbb{Z}$, with metric d inherited from the standard metric on $\mathbb{R}$. The precise class of maps $T: \mathbb{T} \to \mathbb{T}$ which we study is defined as follows.

Definition 1.1. For each $\alpha > 0$, we say that a continuous function $T: \mathbb{T} \to \mathbb{T}$ is an *expanding map of Manneville–Pomeau type α* if it fixes 0, is differentiable with derivative

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greater than 1 in the interval $\mathbb{T} \setminus \{0\}$, and satisfies

$$T'(x) = 1 + \xi x^{\alpha} + o(x^{\alpha}) \quad \text{as } x \to 0^+,$$
$$\liminf_{x \to 1^-} T'(x) > 1$$

for some $\xi > 0$.

The archetypal map T represented by this definition is the *Manneville–Pomeau map* defined by $x \mapsto x + x^{1+\alpha} \mod 1$. Expanding maps of Manneville–Pomeau type are studied in, for example, [**12, 18, 24**].

For each $\gamma \in (0, 1]$, let H_γ denote the space of all γ-Hölder continuous real-valued functions on the circle $\mathbb{T}$, and define $|f|_\gamma = \sup_{x \neq y} |f(x) - f(y)|/d(x, y)^\gamma$ for $f \in H_\gamma$. The set H_γ is a Banach space when equipped with the norm $\|\cdot\|_\gamma$ given by $\|f\|_\gamma := |f|_\infty + |f|_\gamma$. Using a method based on Young towers, S. Branton proved the following.

THEOREM. [**8**] *Let $T : \mathbb{T} \to \mathbb{T}$ be an expanding map of Manneville–Pomeau type $\alpha \in (0, 1)$. Then for every $f \in H_1$ and $\delta \in (0, 1 - \alpha)$ there exists $g \in H_{1-\alpha-\delta}$ such that $f \le g \circ T - g + \beta(f)$.*

We are able to establish the following result.

THEOREM 1. *Let $T : \mathbb{T} \to \mathbb{T}$ be an expanding map of Manneville–Pomeau type $\alpha \in (0, 1)$, and suppose that $\alpha < \gamma \le 1$. Then for every $f \in H_\gamma$ there exists $g \in H_{\gamma-\alpha}$ such that $f \le g \circ T - g + \beta(f)$. In addition, the function g satisfies the functional equation*

$$g(x) + \beta(f) = \max_{Ty=x} [f(y) + g(y)] .$$

Furthermore, we are able to show that Theorem 1 is sharp both in the regularity of f and in the regularity of g.

THEOREM 2. *Let $T : \mathbb{T} \to \mathbb{T}$ be an expanding map of Manneville–Pomeau type $\alpha \in (0, 1)$, and suppose that $0 < \alpha < \gamma \le 1$. Then the following hold:*

(a) *there exists $f \in H_\gamma$ such that if $f \le g \circ T - g + \beta(f)$ for $g \in H_\theta$, then $\theta \le \gamma - \alpha$;*
(b) *there exists $f \in H_\alpha$ such that $f \le g \circ T - g + \beta(f)$ is not satisfied for any continuous function g.*

In a recent article, T. Bousch proved the following theorem, which extends a result of Yuan and Hunt [**25**].

THEOREM. [**4**] *Let $T : X \to X$ be a continuous surjection of a compact metric space. Suppose that for all $f \in H_1$, there exists $g \in H_1$ such that $f \le g \circ T - g + \beta(f)$ and $|g|_1 \le C|f|_1$ for some $C > 0$ independent of f. Suppose also that $\mu \in \mathcal{M}_T$ is a maximizing measure for every element of some non-empty open set $U \subset H_1$. Then μ is supported on a periodic orbit of T.*

We remark that while uniformly expanding dynamical systems have been shown to satisfy the hypotheses of this theorem (see [**3, 11, 22**]), Theorem 2(a) demonstrates that the required hypotheses do not hold for maps of Manneville–Pomeau type.

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2. *Proof of Theorem 1*

We use a fixed-point method that was employed in the work of Bousch [**2, 4**]. We begin with the following lemma.

LEMMA 2.1. *Let T be of Manneville–Pomeau type α, and take $z_1, z_2 \in \mathbb{T}$ with $d(z_1, z_2)$ sufficiently small. Then*

$$d(Tz_1, Tz_2) \geq d(z_1, z_2)(1 + C_0\, d(z_1, z_2)^\alpha)$$

for some constant C_0 that depends only on T.

Proof. We consider separately two cases depending on whether the shortest arc connecting z_1 and z_2 does or does not contain 0.

We begin with the latter case. Choose representatives $a_1, a_2 \in [0, 1)$ of $z_1, z_2 \in \mathbb{T}$, respectively, assuming without loss of generality that $0 \leq a_1 \leq a_2 < 1$. If $d(z_1, z_2)$ is small enough, then

$$\begin{aligned} d(Tz_1, Tz_2) = \int_{z_1}^{z_2} |T'(s)|\, ds &\geq \int_{a_1}^{a_2} 1 + \rho_0 s^\alpha\, ds \\ &\geq (a_2 - a_1) + \rho_1 (a_2 - a_1)^{1+\alpha} = d(z_1, z_2) + \rho_1\, d(z_1, z_2)^{1+\alpha} \end{aligned}$$

for some small $\rho_0, \rho_1 > 0$ not depending on z_1 and z_2. This completes the proof in this case.

Now suppose that 0 lies in the arc connecting z_1 and z_2, with the triple $(z_1, 0, z_2)$ being positively oriented. Arguing as previously, we have $d(Tz_2, 0) \geq d(z_2, 0) + \rho_1 d(z_2, 0)^{1+\alpha}$. Since T has derivative bounded away from 1 in any small interval of the form $(-\delta, 0)$, there is a $\rho_2 > 0$ such that $d(Tz_1, 0) \geq (1 + \rho_2) d(z_1, 0)$ when $d(z_1, 0)$ is small enough. Combining these estimates yields

$$d(Tz_1, Tz_2) = d(Tz_1, 0) + d(0, Tz_2) \geq d(z_1, z_2) + \rho_1\, d(z_2, 0)^{1+\alpha} + \rho_2\, d(z_1, 0).$$

If we take $C_0 = \min\{\rho_1/2^{1+\alpha}, \rho_2/2\}$, then by separating the cases $d(z_1, 0) \geq d(z_2, 0)$ and $d(z_1, 0) \leq d(z_2, 0)$ we obtain

$$\rho_1\, d(z_2, 0)^{1+\alpha} + \rho_2\, d(z_1, 0) \geq C_0\, d(z_1, z_2)^{1+\alpha}$$

for every sufficiently close choice of z_1 and z_2 separated by 0. Combining the above two inequalities completes the proof. □

LEMMA 2.2. *Let T be of Manneville–Pomeau type α, and let $\gamma \in (\alpha, 1]$. Then there exists $C_\gamma > 0$ with the following property: for every $x_1, x_2, y_1 \in \mathbb{T}$ with $Ty_1 = x_1$, we may choose $y_2 \in T^{-1}\{x_2\}$ such that*

$$d(y_1, y_2)^{\gamma-\alpha} + C_\gamma\, d(y_1, y_2)^\gamma \leq d(x_1, x_2)^{\gamma-\alpha}. \tag{1}$$

Proof. Given $x_1, x_2, y_1 \in \mathbb{T}$ with $Ty_1 = x_1$, we claim that there exists $y_2 \in T^{-1}\{x_2\}$ such that

$$d(y_1, y_2)(1 + \rho_3\, d(y_1, y_2)^\alpha) \leq d(x_1, x_2) \tag{2}$$

for some $\rho_3 > 0$ independent of x_1, x_2, y_1. Taking $\rho_4 = (1 + \rho_3)^{\gamma-\alpha} - 1 > 0$, we have $(1 + \rho_3 t)^{\gamma-\alpha} \geq 1 + \rho_4 t$ for all $t \in [0, 1]$. Applying this to (2) yields (1) with $C_\gamma = \rho_4$.

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We now prove the claim. We begin by noting that T expands sufficiently long intervals by a uniform factor: for every $\delta > 0$, there exists $K_\delta > 0$ such that if $d(x_1, x_2) \geq \delta$, then y_2 may be chosen with

$$(1 + K_\delta)\, d(y_1, y_2) \leq d(x_1, x_2).$$

Thus, given some fixed $\delta > 0$, (2) holds for every case in which $d(x_1, x_2) \geq \delta$ by taking $\rho_3 \leq K_\delta$. On the other hand, if $d(x_1, x_2) < \delta$ for some sufficiently small fixed $\delta > 0$, then we may choose $y_2 \in T^{-1}\{x_2\}$ with $d(y_1, y_2) \leq d(x_1, x_2) < \delta$ and apply Lemma 2.1 to obtain

$$d(y_1, y_2)(1 + C_0\, d(y_1, y_2)^\alpha) \leq d(x_1, x_2),$$

so that taking $\rho_3 = \min\{K_\delta, C_0\}$ completes the proof. □

We now prove Theorem 1. Let $\gamma \in (\alpha, 1]$ and define a subset of $C(\mathbb{T})$ by

$$K = \{g \in H_{\gamma-\alpha} : |g|_{\gamma-\alpha} \leq C_\gamma^{-1} |f|_\gamma\},$$

where $C_\gamma > 0$ is as in Lemma 2.2. Let $K_0 = K/\mathbb{R}$, the set of equivalence classes of elements of K modulo addition of a constant. Clearly, K_0 is compact with respect to uniform distance. For each $g \in K$, define $L_f g \in C(\mathbb{T})$ by $(L_f g)(x) = \max_{Ty=x} (f + g)(y)$. We assert that L_f is a continuous transformation of K with respect to uniform distance.

Given $x_1, x_2 \in \mathbb{T}$ and $g \in K$, choose $y_1 \in T^{-1}x_1$ such that $(L_f g)(x_1) = (f + g)(y_1)$. Invoking Lemma 2.2, we may choose $y_2 \in T^{-1}x_2$ such that (1) holds and therefore

$$\begin{aligned}(L_f g)(x_1) - (L_f g)(x_2) &\leq (f + g)(y_1) - (f + g)(y_2)\\ &\leq |f|_\gamma\, d(y_1, y_2)^\gamma + |g|_{\gamma-\alpha}\, d(y_1, y_2)^{\gamma-\alpha}\\ &\leq C_\gamma^{-1} |f|_\gamma\, d(x_1, x_2)^{\gamma-\alpha}.\end{aligned}$$

We conclude that $|L_f g|_{\gamma-\alpha} \leq C_\gamma^{-1}|f|_\gamma$ for all $g \in K$ and therefore $L_f K \subseteq K$. A simple argument shows that $|L_f g_1 - L_f g_2|_\infty \leq |g_1 - g_2|_\infty$ for $g_1, g_2 \in K$ so that L_f is a continuous transformation of K. It follows that L_f induces a continuous transformation of K_0. Hence, by the Schauder–Tychonoff theorem, there exists $h \in K$ such that $L_f h = h$ mod $\mathbb{R}$. Let $b \in \mathbb{R}$ be chosen such that $h(x) = b + \max_{Ty=x}(f + h)(y)$ for all $x \in \mathbb{T}$; a simple argument as in [**2**] shows that $b = \beta(f)$. The proof of Theorem 1 is complete.

3. *Proof of Theorem 2*

In this section we shall take the liberty of using the fundamental domain $[0, 1)$ as a model for $\mathbb{T}$ and treating T as a $[0, 1) \to [0, 1)$ map in the obvious fashion. Let $u_1 = \min\{u \in (0, 1) : Tu = 0\}$ and define a sequence $(u_n)_{n\geq1}$ in $[0, 1)$ by $u_n := \min\{u \in (0, 1) : Tu = u_{n-1}\}$. We require two simple lemmas.

LEMMA 3.1. *There is $C_1 > 1$ such that for all $n \geq 1$,*

$$C_1^{-1} n^{-1-1/\alpha} \leq u_n - u_{n+1} \leq C_1 n^{-1-1/\alpha}$$

and

$$C_1^{-1} n^{-1/\alpha} \leq u_n \leq C_1 n^{-1/\alpha}.$$

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Proof. This follows from the relation $Tu_n - u_n = \xi u_n^{1+\alpha} + o(u_n)^{1+\alpha}$ in a fairly straightforward fashion; see, for instance, [**24**]. □

LEMMA 3.2. *Let $f\colon [0, 1) \to \mathbb{R}$. Assume $f(0) = 0$, and suppose that there is $C > 0$ such that for all $\kappa \in (0, 1)$,*

$$|f(\kappa)| \leq C\kappa^{\gamma_1}$$

and

$$\sup_{\substack{x,y\in[\kappa,1]\\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|} \leq C\kappa^{-\gamma_2},$$

where $\gamma_1, \gamma_2 > 0$ and $\gamma_1 + \gamma_2 \geq 1$. Then f is $\gamma_1/(\gamma_1 + \gamma_2)$-Hölder continuous throughout $[0, 1)$.

Proof. Let $0 \leq x < y < 1$, and let $\lambda = y^{-\gamma_1-\gamma_2}(y - x)$ and $\gamma = \gamma_1/(\gamma_1 + \gamma_2)$. If $\lambda > 1/2$, then $y^{\gamma_1+\gamma_2} < 2(y - x)$ and hence

$$|f(x) - f(y)| \leq |f(x)| + |f(y)| \leq 2Cy^{\gamma_1} < 2^{1+\gamma}C|y - x|^{\gamma}.$$

Otherwise, $y - x = \lambda y^{\gamma_1+\gamma_2} \leq \lambda y \leq y/2$; so $0 < y \leq 2x$ and hence

$$\begin{aligned}|f(x) - f(y)| &\leq Cx^{-\gamma_2}(y - x)^{1-\gamma}(y - x)^{\gamma}\\ &= C\lambda^{1-\gamma}\left(\frac{y}{x}\right)^{\gamma_2}(y - x)^{\gamma} \leq 2^{\gamma-1+\gamma_2}C(y - x)^{\gamma},\end{aligned}$$

as required. □

3.1. *Proof of part (a).* Given $0 < \alpha < \gamma \leq 1$, let $K_\gamma = C_1 \sum_{n=2}^{\infty} n^{-\gamma/\alpha} < \infty$. Define f by $f(x) = x^\gamma$ for all $x \in [0, u_3]$, by $f(x) = -K$ for all $x \in [u_2, u_1]$, and by linear interpolation in the intervals $[u_3, u_2]$ and $[u_1, 1)$ subject to the constraint $\lim_{x\to1} f(x) = 0$ which ensures that f yields a continuous function $\mathbb{T} \to \mathbb{R}$. Note that $f(x) \leq u_k^\gamma$ when $u_{k+1} \leq x \leq u_k$ and that $f \in H_\gamma$.

We claim that $\beta(f) = 0$. Since the Dirac measure δ_0 is invariant and $f(0) = 0$, it is clear that $\beta(f) \geq 0$. By a lemma of Peres [**21**], there exists $x \in \mathbb{T}$ such that $\sum_{j=0}^{n-1} f(T^j x) \geq n\beta(f)$ for all $n \geq 0$; so to prove that $\beta(f) \leq 0$, it is sufficient to show that for each $x \in [0, 1]$ we may find $v(x) > 0$ such that $\sum_{j=0}^{v(x)-1} f(T^j x) \leq 0$.

If $x = 0$ or $x \in [u_2, 1)$, then clearly we may take $v(x) = 1$. Otherwise, we have $x \in [u_{r+1}, u_r]$ for some $r \geq 2$. Applying Lemma 3.1, we obtain

$$\sum_{j=0}^{r} f(T^j x) \leq \sum_{j=0}^{r-2}(T^j x)^\gamma - K \leq \sum_{k=2}^{r} u_k^\gamma - K \leq C_1 \sum_{k=2}^{\infty} k^{-\gamma/\alpha} - K = 0,$$

so that taking $v(x) = r + 1$ proves the claim.

Now suppose that $f \leq g \circ T - g + \beta(f)$, where $g \in H_\theta$. For every $n > 0$ and $r \geq 3$, we have

$$g(u_{n+r}) + \sum_{j=0}^{n-1} f(T^j u_{n+r}) \leq g(T^n u_{n+r})$$

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and hence

$$g(u_r) \geq \sum_{k=r+1}^{r+n} f(u_k) + g(u_{n+r}) \geq C_1^{-1} \sum_{k=r+1}^{r+n} k^{-\gamma/\alpha} + g(u_{n+r}).$$

Taking the limit as $n \to \infty$ gives

$$g(u_r) \geq C_1^{-1} \sum_{k=r+1}^{\infty} k^{-\gamma/\alpha} + g(0) \geq \tilde{C} r^{1-\gamma/\alpha} + g(0),$$

and therefore

$$\tilde{C} r^{-1-\gamma/\alpha} \leq |g(0) - g(u_r)| \leq |g|_\theta u_r^\theta \leq |g|_\theta C_1^\theta r^{-\theta/\alpha}$$

for every $r \geq 3$. We deduce that $\theta \leq \gamma - \alpha$. □

3.2. *Proof of part (b).* Define $f(0) = 0$, $f(x) = 0$ for all $x \in [u_1, 1)$ and, for each $n \geq 0$,

$$f(u_{2^{4n}}) = f(u_{2^{4n+2}}) = 0,$$
$$f(u_{2^{4n+1}}) = -2^{-4n},$$
$$f(u_{2^{4n+3}}) = \tau 2^{-4n},$$

where $\tau \in (0, 1)$ is a real number to be fixed later. Extend f to the whole of $[0, 1)$ by interpolating linearly in each interval $[u_{2^{4n+k+1}}, u_{2^{4n+k}}]$.

We will show that f is α-Hölder. Suppose that $u_{2^{4n+4}} \leq \kappa \leq u_{2^{4n}}$ for some $n \geq 0$; then

$$|f(\kappa)| < 2^{-4n} \leq C_1^\alpha u_{2^{4n}}^\alpha \leq C_1^\alpha \kappa^\alpha. \tag{3}$$

We must estimate the Lipschitz norm of f in the interval $[\kappa, 1)$. To do this, we require the simple lower bound

$$\begin{aligned} u_{2^{r+1}} - u_{2^r} &= \sum_{\ell=0}^{2^r-1} u_{2^r+\ell+1} - u_{2^r+\ell} \geq \sum_{k=2^r}^{2^{r+1}-1} C_1^{-1} k^{-1-1/\alpha} \\ &\geq \tilde{C}(2^{-r/\alpha} - 2^{-(r+1)/\alpha}) \geq \tilde{C} 2^{-r/\alpha} \end{aligned}$$

for all $r > 0$, where we have used Lemma 3.1. It follows that when $u_{2^{4n+4}} \leq \kappa \leq u_{2^{4n}}$, the gradient of f in $[\kappa, 1)$ is bounded by

$$\sup_{\substack{0 \leq k \leq n \\ 0 \leq \ell < 4}} \frac{2^{-4k}}{|u_{2^{4k+\ell+1}} - u_{2^{4k+\ell}}|} \leq \sup_{\substack{0 \leq k \leq n \\ 0 \leq \ell < 4}} \frac{2^{-4k}}{\tilde{C} 2^{-(4k+\ell)/\alpha}} = \tilde{C} 2^{-4k+4k/\alpha} \leq \tilde{C} \kappa^{\alpha-1}. \tag{4}$$

Combining estimates (3) and (4) with Lemma 3.2, we deduce that $f \in H_\alpha$.

We next compute $\beta(f)$. Since $f(0) = 0$ and the Dirac measure δ_0 is T-invariant, we have $\beta(f) \geq 0$. To prove that $\beta(f) = 0$, we proceed as in part (a) by showing that for each $x \in [0, 1)$, there is $v(x) > 0$ such that $\sum_{j=0}^{v(x)-1} f(T^j x) \leq 0$.

If $x \geq u_2$ or $x = 0$ or $u_{2^{4n+2}} \leq x \leq u_{2^{4n}}$ for some $n > 0$, then $f(x) \leq 0$ and we may take $v(x) = 1$. We therefore restrict our attention to the case in which $u_{2^{4n+4}} < x < u_{2^{4n+2}}$ for some $n \geq 0$. Assuming this, suppose that

$$u_{2^{4n+2}+k+1} \leq x \leq u_{2^{4n+2}+k},$$

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where $0 \le k < 2^{4n+4} - 2^{4n+2}$. We choose $v(x) = k + 2^{4n+1} + 2$. First we note that

$$\sum_{j=0}^{k} f(T^j x) \le \tau k 2^{-4n} \le 12\tau. \tag{5}$$

Using the monotonicity of f in $[u_{2^{4n+1}}, u_{2^{4n}}]$, we obtain

$$\begin{aligned}
\sum_{j=k+1}^{k+2^{4n+1}+1} f(T^j x) &\le \sum_{\ell=0}^{2^{4n+1}} f(u_{2^{4n+1}+\ell}) = -\sum_{\ell=1}^{2^{4n+1}} 2^{-4n} \frac{|u_{2^{4n+1}} - u_{2^{4n+1}+\ell}|}{|u_{2^{4n+1}} - u_{2^{4n+2}}|} \\
&\le -\sum_{\ell=1}^{2^{4n+1}} 2^{-4n} u_{2^{4n+1}}^{-1} (u_{2^{4n+1}} - u_{2^{4n+1}+\ell}) \\
&\le -C_1^{-1} 2^{-1/\alpha - 4n + 4n/\alpha} \sum_{\ell=1}^{2^{4n+1}} \sum_{j=0}^{\ell-1} (u_{2^{4n+1}+j} - u_{2^{4n+1}+j+1}) \\
&\le -C_1^{-2} 2^{-1/\alpha - 4n + 4n/\alpha} \sum_{\ell=1}^{2^{4n+1}} \ell 2^{-(4n+2)(1+1/\alpha)} \\
&\le -\frac{1}{C_1^2 2^{2+3/\alpha}} 2^{-8n} \sum_{\ell=1}^{2^{4n+1}} \ell \le -\frac{1}{C_1^2 2^{5+2/\alpha}} = -\varepsilon < 0,
\end{aligned}$$

say, where we have twice used Lemma 3.1. Combining this estimate with (5), we deduce that $\sum_{j=0}^{v(x)-1} f(T^j x) \le \max\{0, 12\tau - \varepsilon\}$ for each $x \in [0, 1)$; thus, if τ is taken smaller than $\varepsilon/12$, then $\beta(f) = 0$.

Our final task is to show that the relation $f \le g \circ T - g + \beta(f)$ is impossible for continuous g. Following the method of the preceding estimate, for each $n > 0$ we have

$$\begin{aligned}
\sum_{\ell=2^{4n+2}}^{2^{4n+3}} f(u_\ell) &\ge \tau \sum_{\ell=1}^{2^{4n+2}} 2^{-4n} \frac{|u_{2^{4n+2}} - u_{2^{4n+2}+\ell}|}{|u_{2^{4n+2}} - u_{2^{4n+3}}|} \\
&\ge \tau \tilde{C} 2^{-4n+4n/\alpha} \sum_{\ell=1}^{2^{4n+2}} \sum_{j=0}^{\ell-1} (u_{2^{4n+2}+j} - u_{2^{4n+2}+j+1}) \\
&\ge \tau \tilde{C} 2^{-8n} \sum_{\ell=1}^{2^{4n+2}} \ell \ge \delta_\tau > 0,
\end{aligned}$$

say. Suppose now that $f \le g \circ T - g + \beta(f)$ is satisfied; then for each $n > 0$ we have

$$g(u_{2^{4n+2}}) \ge g(u_{2^{4n+3}}) + \sum_{j=0}^{2^{4n+3}-2^{4n+2}} f(T^j u_{2^{4n+3}}) \ge g(u_{2^{4n+3}}) + \delta_\tau.$$

If g is continuous at 0, letting $n \to \infty$ then yields

$$g(0) \ge g(0) + \delta_\tau > g(0),$$

which is a contradiction. □

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Acknowledgements. The author would like to thank S. Branton, O. Jenkinson, M. Nicol and P. Thieullen for helpful conversations. He also thanks O. Jenkinson for organising the School and Workshop on Ergodic Optimization at Queen Mary, University of London in November 2006, at which many of these discussions took place.

The author is grateful to an anonymous (and patient!) referee for suggesting some significant simplifications. The proof of Theorem 1, in particular, has benefited substantially from the referee's advice.

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